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PRECIPITATION OF A CLOUD OF INTERACTING PARTICLES FOLLOWED BY "DUSTING" SOURCE FORMATION ON ACCOUNT OF ATMOSPHERIC DIFFUSION

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SUMMARY

This paper studies the distribution of concentration on the Earth's surface in case of precipitation of a cloud of interacting particles formed as a result of action of a point source, when the concentration is higher than a certain constant subject to experimental determination. Simple formulas are given for the calculation of the turbulent diffusion coefficient by the experimentally measured distance from the point of admixture dropping to the points corresponding to diffusion minimum and maximum of the windward surface concentration.

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We shall investigate the distribution of concentration on the surface of the Earth in the case when the cloud precipitation of interacting particles due to the action of a point source, provided the concentration q is $> q_0$, where q_0 is a certain constant subject to experimental determination. We shall not dwell here upon the causes inducing this interaction, for they may in each particular case be the object of special study. Because of the turbulent diffusion taking lace in time and space (and possibly also on other counts), the cloud is washed off from surface only, which results in particle concentration in the part washed away lower than q_0. The interaction of particles is disrupted in this part of the cloud and the particles, separated from the cloud, precipitate with the Stokes velocity w. It is assumed that the cloud as a whole settles with a velocity V.

The particles having separated from the cloud form a "dusting" source. The intensity of the source

$$Q = Q_0 f(t)\delta(x)\delta(y)\delta[z - h(1 - t/T) \times U(1 - t/T)]$$

where U(t) is a unitray function; $T = h/V$; $f(t) = \beta t e^{-\alpha t}$; xy is the Earth's plane; z is the vertical coordinate and t is the time.

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* OSAZHDENIYE OBLAKA VZAIMODEYSTVUYUSHCHIKH CHASTITS I OBRAZOVANIYE PRI ETOM "PYLYASHCHEGO" ISTOCHNIKA V REZUL'TATE ....

The function Q determines the character of source's intensity variation as a function of its position in space and time. The quantity β will be determined from the condition

$$\beta \int_0^\infty \tau e^{-\alpha\tau} d\tau = 1, \quad \beta = \alpha^2.$$

The number of particles diffusing from the cloud during the time T will be determined from the expression

$$Q_0\alpha^2 \int_0^T \tau e^{-\alpha\tau} d\tau = Q_0\left[1 - \alpha e^{-\alpha T}\left(T + \frac{1}{\alpha}\right)\right].$$

The function f (t) is chosen so as to have a maximum at a certain point of the time interval $(0, \infty)$ and be zero at the ends of this interval.

We resolve the turbulent diffusion equation of a heavy admixture

$$\partial q / \partial t = K_x \partial^2 q / \partial x^2 + K_y \partial^2 q / \partial y^2 + K_z \partial^2 q / \partial z^2 - u \, \partial q / \partial x + w \, \partial q / \partial z + Q. \tag{1}$$

Here u is the horizontal component of wind velocity; w is the gravitational rate of admixture precipitation; K_x, K_y, K_z are the respective turbulent diffusion coefficients of the admixture.

The equation (1) is resolved for the following initial and boundary conditions: at t = 0, q = 0; at z = 0, q = 0; at $\sqrt{x^2 + y^2 + z^2} \to \infty$, q = 0.

All the coefficients of the equation (1) are assumed to be constant. In such a case its solution will be written in the following manner:

$$q = \frac{Q_0\alpha^2}{8(\pi^3 K_x K_y K_z)^{1/2}} \int_0^t F(t, \tau, x, y, z)\, d\tau \qquad \text{at} \qquad t < T; \tag{2$^{\text{а}}$}$$

$$q = \frac{Q_0\alpha^2}{8(\pi^3 K_x K_y K_z)^{1/2}} \int_0^T F(t, \tau, x, y, z)\, d\tau +$$

$$+ \frac{Q'\alpha^2}{8(\pi^2 K_x K_y)^{1/2} K_z^{3/2}} \int_T^t \frac{z f(\tau)}{(t-\tau)^{5/2}} \exp\left[-\frac{[x - u(t-\tau)]^2}{4K_x(t-\tau)} - \frac{y^2}{4K_y(t-\tau)} - \right.$$

$$\left. - \frac{w^2(t-\tau)}{4K_z} - \frac{wz}{2K_z} - \frac{z^2}{4K_z(t-\tau)}\right] d\tau \qquad \text{at} \qquad t > T, \tag{2$^{\text{б}}$}$$

where

$$Q' = \lim_{h \to 0} (Q_0 h),$$

$$F(t, \tau, x, y, z) = \frac{f(\tau)}{(t-\tau)^{3/2}} \exp\left[-\frac{[x - u(t-\tau)]^2}{4K_x(t-\tau)} - \frac{y^2}{4K_y(t-\tau)} - \right.$$

$$\left. - \frac{w^2(t-\tau)}{4K_z} - \frac{w[z - h(1 - T^{-1}\tau)]}{2K_z}\right]\left\{\exp\left[-\frac{[z - h(1 - T^{-1}\tau)]^2}{4K_z(t-\tau)}\right] - \right.$$

$$\left. - \exp\left[-\frac{[z + h(1 - T^{-1}\tau)]^2}{4K_z(t-\tau)}\right]\right\}.$$

The second addend in the expression $(2^{\text{б}})$ approaches zero as a consequence of the condition of admixture absorption on the surface z = 0. The obtained solution is valid at arbitrary assigning h (t) as a function of time. With such a statement of the problem the distribution of concentration at the Earth's surface has three extreme points (two maxima and one minimum).

In this paper we have given simple formulas for the calculation of turbulent diffusion coefficients by the experimentally measured distances from the point of admixture drop to points corresponding to minimum and diffusion maximum of concentration surfaces (in the wind direction).

We shall perform in (2^a) and ($2^б$) the variable substitution $t - \tau = \xi$, and thus have

$$q = \int_0^t F(t, \xi, x, y, z)\, d\xi, \quad t \leqslant T; \tag{3^a}$$

$$q = \int_{t-T}^{t} F(t, \xi, x, y, z)\, d\xi, \quad t > T. \tag{$3^б$}$$

For the determination of particle concentration on the Earth's surface q^*, we shall compute the integral

$$q^* = \int_0^\infty K_z \frac{\partial q}{\partial z}\Big|_{z=0} dt. \tag{4}$$

Inasmuch as the function $q(z, t)$ is defined in the interval $0 \leqslant z \leqslant \infty$, and the subintegral expression as a function of z and ξ is continuous in the semi-infinite band $0 \leqslant \xi \leqslant t$ and has in that region a partial derivative over z, z, the formula

$$q^* = H\Big[\int_0^\infty \exp(-C) \int_0^t F_1(\xi, t)\, d\xi\, dt - \int_T^\infty \exp(-C) \int_0^{t-T} F_1(\xi, t)\, d\xi\, dt\Big], \tag{5}$$

where
$$F_1(\xi, t) = \frac{t - \xi - T^{-1}(t-\xi)^2}{\xi^{5/2}} \exp\Big[-\frac{A}{\xi} - B\xi\Big],$$

$$H = Q_0 \alpha^2 h / 8(\pi^3 K_x K_y K_z)^{1/2}, \quad A = x^2/4K_x + y^2/4K_y + h^2(1 - T^{-1}t)^2/4K_z,$$

$$B = u^2/4K_x + \frac{(w-V)^2}{4K_z} - \alpha, \quad C = -xu/2K_x - h(w - V)(1 - T^{-1}t)/2K_z + \alpha t.$$

takes place in the interval $[0, \infty]$ for any z.

Let us compute the inner integral in (5). According to (1), it is

$$\int_0^u x^{\nu-1}(u-x)^{\mu-1} e^{-\beta/x} dx = \beta^{(\nu-1)/2} u^{(2\mu+\nu-1)/2} \exp\Big(\frac{\beta}{2u}\Big) \Gamma(\mu) W_{(1-2\mu-\nu)/2,\, \nu/2}\Big(\frac{\beta}{u}\Big).$$

We shall expand $\exp(-B\xi)$ in Taylor series; after integration we shall have

$$q^* = H \int_0^\infty \exp(-C) \exp\Big(-\frac{A}{2t}\Big) A^{-5/4} t^{3/4},$$

$$\sum_{n=0}^\infty \frac{(-\beta)^n}{n!} (At)^{n/2} \Big[\Gamma(2) W_{-(3/4+n/2),\, (1/4+n/2)}\Big(\frac{A}{t}\Big) - T^{-1} t W_{-(5/4+n/2),\, (-1/4+n/2)}\Big(\frac{A}{t}\Big)\Big] dt,$$

where $W_{\lambda,\mu}$ is a Whittaker function; Γ is a gamma-function.

It is well known that for great values of the argument the Whittaker function has the form

$$W_{\lambda,\mu}(z) \sim e^{-z/2} z^{\lambda}\left(1+\sum_{k=1}^{\infty}\frac{[\mu^2-(\lambda-1/2)^2]\ldots[\mu^2-(\lambda-k+1/2)^2]}{k!\, z^k}\right). \tag{6}$$

For sufficiently great x the condition $A/t \gg 1$ is satisfied. In such a case, utilizing the asymptotic expansion of the Wittaker function (6), valid for great values of the argument, we shall obtain

$$q^* \simeq H\Gamma(2)\int_0^{\infty}\exp\left(-d-\frac{a}{t}-bt\right)\exp\left[-\frac{Bh^2}{4K_zT^2}(t-T)^2\right]t^{5/2}A^{-2}\,dt -$$
$$-\int_T^{\infty}\exp(-C)\int_0^{t-T}F_1(\xi,t)\,d\xi\,dt, \tag{*}$$

where $d=-xu/2K_x-\frac{h(w-V)}{2K_z}-\frac{Uh}{2K_z}+B\left(\frac{x^2}{4K_z}+\frac{y^2}{4K_y}\right)$, $a=x^2/4K_x+$ $+y^2/4K_y+h^2/4K_z$, $b=\frac{wV}{2K_z}-\frac{V^2}{4K_z}+\alpha$.

The second exponent, standing under the first integral, will be a Delta function under the condition $Bh^2/4K_z \gg 1$, it is idemponent to the condition $u^2h^2/16K_xK_z \gg 1$. For a free atmosphere, that is for $h > 1000$ m, it is fulfilled. The first exponent standing under the integral is a slowly-varying function. Therefore, we may write

$$q^* \simeq H\Gamma(2)A^{-2}\exp\left[-d-\frac{a}{b}-bT\right]\int_0^{\infty}\exp\left[-\frac{Bh^2}{4K_zT^2}(t-T)^2\right]dt-H\int_T^{\infty}\exp(-C)\int_0^{t-T}F_1(t,\xi)\,d\xi\,dt. \tag{7}$$

The integral

$$\int_T^{\infty}\exp(-C)\int_0^{t-T}F_1(t,\xi)\,d\xi\,dt \leqslant \int_1^{\infty}\exp(-C)\int_0^{t}F_1(t,\xi)\,d\xi\,dt.$$

That is why

$$q^*=\gamma(T)\Gamma(2)A^{-2}\exp\left[-d-\frac{a}{T}-bT\right]T^{5/2}\int_0^{\infty}\exp\left[-\frac{Bh^2}{4K_zT^2}(t-T)^2\right]dt$$

and $\qquad 1/2<\gamma(T)\leqslant 1$ at $0<T\leqslant\infty$.

The first addend of (7) represents the first term of the expansion (*) in Taylor series in the vicinity of the point $t = T$. Let us estimate the subseauent terms of this expansion. To that effect we shall expand $\exp[-a/t-bt]$ in Taylor series in the vicinity of the point $t = T$ and integrate the expression

$$\sum_{n=1}^{\infty}\int_0^{\infty}(t-T)^n\exp\left[-\frac{Bh^2}{4K_zT^2}(t-T)^2\right]dt.$$

Taking into account that the time T is a sufficiently great quantity (>10 s), we shall obtain the condition for which all the subsequent terms of the expansion will be small:

$$\frac{1}{4}\frac{x^2}{h^2}\frac{V}{u}\sqrt{\frac{K_z}{K_x}}\leqslant 1.$$

Formula (7) can be applied for the determination of the values of the coefficients K_x and K_z by experimental measurements of the coordinates of the extreme points of the function q^*. Let us determine the extremal points of the function q^*. Equating the derivative $\partial q^* / \partial x$ to zero we obtain $x_{1,2} = \pm\infty$, and also

$$\tilde{x} = x(BT+1)/uT, \quad \tilde{b}_1 = 8K_x(BT+1)/u^2T + \tilde{y}^2K_x/K_y, \quad \tilde{b}_0 = \tilde{y}^2K_x/K_y, \tilde{y} = y(BT+1)/uT.$$

where

$$\tilde{x}^3 - \tilde{x}^2 + \tilde{b}_1\tilde{x} - \tilde{b}_0 = 0, \tag{8}$$

Postulating $y = 0$, we obtain $\tilde{b}_0 = 0$,

$$\tilde{x}(\tilde{x}^2 - \tilde{x} + \tilde{b}_1) = 0, \tag{9}$$

consequently

$$\tilde{x}_3 = 0, \quad \tilde{x}_4 = {}^1/_2(1 - \sqrt{1 - 4\tilde{b}_1}), \quad \tilde{x}_5 = {}^1/_2(1 + \sqrt{1 - 4\tilde{b}_1}). \tag{10}$$

The function q^* has at the points $\tilde{x}_3$ and $\tilde{x}_5$ a maximum and at the point $\tilde{x}_4$ a minimum. In fact, the first maximum is located at the point $x_3 = uT$. Such a discrepancy in the position of x_3 was obtained because of the condition when deriving the equation (8).

In the case $\tilde{b}_1 = {}^1/_4$ the radicals of $\tilde{x}_4$ and $\tilde{x}_5$ coincide. At the point the function q^* has an inflexion point. The maximum of the function q^*, forming on account of particle diffusion, vanishes. A maximum exists only at the point $x_3 = uT$. It follows from (10) that at $\tilde{b}_1 \leqslant {}^1/_4$ there exist a minimum and a remote maximum of the function q^*.

In the assumption $y = 0$

$$\tilde{b}_1 = 8K_x(BT+1)/u^2T$$

or

$$\tilde{b}_1 = 2 + \frac{2(w-V)^2}{u^2}\frac{K_x}{K_z} + \frac{K_x}{u^2}\left(\frac{1}{T} - \alpha\right) \leqslant \frac{1}{4}. \tag{11}$$

If the maximum of the function f (t) is located on the Earth's surface, i.e., $\alpha = 1/T$, the condition (11) is not fulfilled. The simplest way of determining the quantities K_x and K_z is by utilizing the properties of the roots of the quadratic equation

$$\tilde{x}^2 - \tilde{x} + \tilde{b}_1 = 0,$$

$$x_4 + x_5 = \frac{Tu}{BT+1}, \quad x_4x_5 = \tilde{b}_1\left(\frac{Tu}{BT+1}\right)^2. \tag{12}$$

From the system (12) we determine the coefficients K_x and K_z,

$$K_x = x_4x_5u / 8(x_4 + x_5),$$

$$K_z = \frac{1}{4}\,\frac{(w-V)^2}{\alpha - 1/T + u(x_4x_5 - 2(x_4+x_5)^2)/(x_4+x_5)\,x_4x_5}. \tag{13}$$

Formulas (13) allow to compute the values of the turbulent diffusion coefficients by the measurements of the position of the extreme points of the function q^*.

Let us investigate the following particular cases :

a) $T = 0$ ($V \to \infty$). Here the condition (11) is not fulfilled.

b) $T \to \infty$ ($V = 0$). There is in this case a source situated at the height h, with an intensity varying in time.

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**** THE END ****

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